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LETTER TO THE EDITOR

Discrete mass spectrum of $Z(N)$ spin systems perturbed by a thermal field

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Abstract. We calculate numerically the lowest masses in the discrete mass spectrum of the $Z(N)$ Fateev–Zamolodchikov spin model perturbed by a thermal operator. The mass ratios obtained are in agreement with those derived from the S-matrix introduced by Köberle and Swieca.

During the last few years the hypothesis of conformal invariance has led to a considerable advance in the description and understanding of critical phenomena in the two-dimensional arena (see [1] and [2] for a review). Recently, some methods have been derived [3–7] which permit us to obtain information about the off-critical theory in the neighbourhood of a critical point. In particular, in a series of papers, Zamolodchikov [5–7] suggested that if the conformal invariant theory (massless) is perturbed by a suitable chosen relevant scaling field the massive off-critical theory may have non-trivial conserved charges and might even be integrable. By a bootstrap approach the complete mass spectrum of the off-critical theory can be calculated.

Numerical calculations of the mass spectrum of the Ising model (conformal anomaly $c=\frac{1}{2}$) [8, 9], Blume–Capel [10] or tricritical Ising model ($c=\frac{7}{10}$) [11, 12] and Ashkin–Teller model ($c=1$) [13] gave support to the above results.

In this letter we calculate numerically the discrete mass spectrum of the $Z(N)$ Fateev–Zamolodchikov model [14] perturbed by a thermal operator (energy operator). Instead of working with the transfer matrix we will work with their associated quantum Hamiltonian [15]

$$H_N(\delta)=-\frac{1}{N}\sum_{i=1}^{L}\sum_{n=1}^{N-1}\{[S(i)S^{+}(i+1)]^{n}+(1+\delta)R^{n}(i)\}/\sin(\pi n/N) \qquad (1)$$

where L is the lattice size and $S(i)$, $R(i)$ are 4×4 matrices satisfying their $Z(N)$ algebra

$$[S(i), R(j)]=[S(i), S(j)]=[R(i), R(j)]=0 \qquad i\neq j \qquad (2a)$$

$$S(i)R(j)=\exp(\mathrm{i}2\pi/N)R(i)S(j) \qquad R^{N}(i)=S^{N}(i)=1. \qquad (2b)$$

Earlier numerical results [16] show us that the infinite system ($L\to\infty$) is critical at $\delta=\delta_c=0$, being the critical fluctuations governed by the $Z(N)$ parafermionic quantum field theory introduced by Zamolodchikov and Fateev [17]. The conformal anomaly of these theories has the value $c=2(N-1)/(N+2)$ and the cases $N=2$ and $N=3$ correspond to the Ising and three-state Potts model.

The thermal perturbation ($\delta\neq 0$) in (1) does not destroy the global $Z(N)$ symmetry and because (1) at $\delta=0$ has an infinite number of conserved charges we expect that their associated scattering S-matrix is that derived some years ago by Köberle and Swieca [18] where the mass ratios are given by

$$\frac{m_i}{m_1}=\frac{\sin(\pi i/N)}{\sin(\pi/N)} \qquad i=1,2,\ldots,N-1. \tag{3}$$

The correlation length of the infinite system (1), for $\delta\neq 0$, diverges as $\delta^{1/(2-X_\varepsilon)}$ when $\delta\to 0$, where $X_\varepsilon=4/(N+2)$ is the dimension of the energy (thermal) operator [17]. In order to calculate numerically the mass spectrum and test the conjecture (3) we apply the numerical scheme followed by Sagdeev and Zamolodchikov [9] in the study of the Ising model in an external magnetic field. According to this scheme in the asymptotic regime, where $\delta\to 0$ and $L\to\infty$, with

$$X=\delta^{1/(2-X_\varepsilon)}L=\delta^{(N+2)/2N}L \tag{4}$$

kept fixed, the zero-momentum eigenenergies $E_K(\delta, L)$ $(K=0,1,2,\ldots)$, for $N>2$, should behave as

$$E_K(\delta, L)=e_\infty L+\delta^{(N+2)/2N}F_K(X)+\delta^{(N+6)/2N}G_K(X) + \delta^{(N+10)/2N}H_K(X)+\delta^{(N+14)/2N}I_K(X)+\ldots \tag{5}$$

where e_∞ is the ground-state energy per particle of the infinite system at $\delta=\delta_c=0$. The masses are obtained from the large-X behaviour of the functions $F_K(X)$, i.e.

$$m_K\sim F_K(X)-F_0(X) \tag{6}$$

where $F_0(X)$ is the function associated in (5) with the ground-state energy. In order to derive (5) we have initially analysed the finite-size corrections of (1) at $\delta=0$ and verified that the most important corrections are due to the $Z(N)$ neutral operator with dimension $X_{\varepsilon\varepsilon}=2+X_\varepsilon=(8+2N)/(N+2)$.

The Hamiltonian (1) with periodic boundary conditions commutes with the $Z(N)$ charge operator $\tilde{Q}$ defined by

$$\exp(\mathrm{i}2\pi\tilde{Q}/N)=\prod_{j=1}^{L}R(j) \tag{7}$$

for arbitrary values of δ. Consequently in the basis where the $R(i)$ operators are diagonal the Hilbert space is separated into N disjoint sectors labelled by the eigenvalues of $\tilde{Q}$ $(q=0,1,\ldots,N-1)$. The ground state belongs to the $q=0$ sector, the first excited state to the sector with $q=1$ and the sectors with $\tilde{Q}=q$ and $\tilde{Q}=N-q$ $(q=1,2,\ldots,N-1)$ are degenerated. These sectors can be further block diagonalized according to the eigenvalues of the translation operator in the lattice (linear momentum).

We have computed numerically, for small lattice sizes, the low zero-momentum eigenstates of the $Z(N)$ Hamiltonian (1), with $N=3,\ldots,7$ and $\delta>0$, where the model is in its disordered (paramagnetic) regime. We then use relation (5), for some values of X, in order to calculate the functions $F_K(X)$. Let us denote by $F_K^q(X)$ the function related to the K zero-momentum state $(K=0,1,2,\ldots)$ in the sector with charge $\tilde{Q}=q$ and obtain, assuming (5) exact for lattices sizes, $L-3$, $L-2$, $L-1$ and L. The mass ratios are then obtained in the asymptotic regime $L\to\infty$ and $X\to\infty$ of the finite-size sequences

$$R_K^q(X, L)=\frac{F_K^q(X, L)-F_0^0(X, L)}{F_0^1(X, L)-F_0^0(X, L)}\to\frac{m_K^q}{m_0^1} \tag{8}$$

where m_K^q are the associated masses and $m_0^1=m_1$ is the lightest one.

Table 1. The mass-ratio estimators $R_k^q(X, L)$ defined in (8) for the $Z(N)$ models ($N = 3, \ldots, 7$). The conjectured values are obtained by combining the ratios given by (2).

Model	L	X	R_1^0	R_1^1	R_0^2	R_1^2	R_0^3
$Z(3)$	13	4	1.998 994	2.149 689	—	—	—
	13	6	2.001 497	2.072 017	—	—	—
	13	8	2.001 880	2.042 343	—	—	—
	13	10	2.001 287	2.019 803	—	—	—
	Conjectured		2	2	—	—	—
$Z(4)$	10	4	2.007 758	2.526 061	1.369 576	2.622 030	—
	10	6	1.998 051	2.430 513	1.405 939	2.250 722	—
	10	8	2.000 372	2.420 605	1.415 346	2.125 526	—
	10	10	2.002 456	2.420 383	1.419 176	2.072 210	—
	Conjectured		2	2.414 213	1.414 213	2	—
$Z(5)$	9	4	1.990 094	2.585 135	1.545 371	2.789 032	
	9	6	1.997 214	2.591 925	1.600 500	2.323 987	—
	9	8	2.003 701	2.612 732	1.617 361	2.164 232	—
	9	10	2.006 502	2.630 395	1.630 808	2.095 762	—
	Conjectured		2	2.618 033	1.618 033	2	—
$Z(6)$	8	4	1.979 226	2.620 715	1.624 995	2.976 190	1.851 969
	8	6	1.998 833	2.659 639	1.683 943	2.404 216	1.944 041
	8	8	2.010 641	2.701 922	1.713 606	2.200 243	1.972 448
	8	10	2.014 101	2.744 378	1.749 150	2.116 954	2.015 411
	Conjectured		2	2.732 050	1.732 050	2	2
$Z(7)$	7	4	1.961 263	2.596 376	1.634 638	3.155 652	1.990 213
	7	6	2.000 031	2.625 581	1.693 112	2.464 406	2.100 318
	7	8	2.020 937	2.726 899	1.757 506	2.233 377	2.180 908
	7	10	2.012 517	2.826 107	1.830 612	2.145 080	2.283 557
	Conjectured		2	2.801 937	1.801 937	2	2.246 979

Table 2. The discrete mass spectrum in the charge sector $\tilde{Q}$ (0, 1, ...) of the $Z(N)$ models ($N = 3, \ldots, 7$). The masses are given by $m_i = m_1 \sin(\pi i/N)/\sin(\pi/N)$ ($i = 1, 2, \ldots$).

Model	$\tilde{Q}=0$	$\tilde{Q}=1$	$\tilde{Q}=2$	$\tilde{Q}=3$
$Z(3)$	$2m_2$	$m_1, 2m_1$	—	—
$Z(4)$	$2m_2$	m_1, m_1+m_2	$m_2, 2m_1$	—
$Z(5)$	$2m_1$	m_1, m_1+m_2	$m_2, 2m_1$	—
$Z(6)$	$2m_1$	m_1, m_1+m_2	$m_2, 2m_1$	—
$Z(7)$	$2m_1$	m_1, m_1+m_2	$m_2, 2m_1$	m_3

In table 1 we present for the largest lattice sizes, compatible with our computational facilities, the estimators $R_k^q(X, L)$ for $N = 3$ ($L = 13$), $N = 4$ ($L = 10$), $N = 5$ ($L = 9$), $N = 6$ ($L = 8$) and $N = 7$ ($L = 7$). We also show in this table the conjectured mass ratios given in (2). These results together with our numerical analysis of the asymptotic ($L \to \infty$, $X \to \infty$) values of the finite-size sequences (8) clearly imply that the mass ratios occurring in the $Z(N)$ Hamiltonian (1) are given by (2), thus indicating the massive field theory introduced in [18] as its underlying field theory. Finally in table 2 we summarize our results for the lowest discrete masses in the several charge sectors.

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